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Fractional statistics and duality: strong tunneling behavior of edge states of quantum Hall liquids in the Jain sequence

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While the values for the fractional charge and fractional statistics coincide for fractional Hall (FQH) states in the Laughlin sequence, they do not for more general FQH states, such as those in the Jain sequence. This mismatch leads to additional phase factors in the weak coupling expansion for tunneling between edge states which alter the nature of the strong tunneling limit. We show here how to construct a weak-strong coupling duality for generalized FQH states with simple unreconstructed edges. The correct dualization of quasiparticles into integer charged fermions is a consistency requirement for a theory of FQH edge states with a simple edge. We show that this duality also applies for weakly reconstructed edges.

The existence of excitations with fractional statistics, both Abelian and non-Abelian, is one of the most startling and unavoidable predictions of the theory of the fractional quantum (FQH) Hall effect [1, 2, 3]. Information on the quasiparticle statistics can in principle be extracted using tunneling and interference experiments. Consequently, a number of proposals for the measurement of fractional statistics of FQH fluids have been made in recent years. These schemes consist of tunneling interferometers which make use of the properties of the edge state tunneling in the FQH effect [4, 5, 6, 7, 8, 9]. Two recent experiments, using an interferometer in which $\nu = 1/3$ quasiparticles can enclose an island with filling fraction $\nu = 2/5$, reported observations consistent with fractional quasiparticle statistics and with chiral Luttinger liquid dependence of the thermal dephasing of the Aharonov-Bohm oscillations [10].

In most theoretical proposals for detecting fractional statistics, calculations are carried out at the lowest orders in perturbation theory in the tunneling amplitude at a point contact. For the Laughlin states the problem of a single point contact can be fully solved under the assumption that the edge remains sharp (i.e. it is unreconstructed). As the tunneling amplitude increases, the dynamics of the point contact evolves from being dominated by quasiparticle tunneling at weak coupling to a strong coupling regime in which the fluid is pinched off and electron tunneling dominates[11, 12]. This process is embodied by a powerful electron-quasiparticle duality[12, 13] which also encodes the information on the quantum numbers of the excitations. Since for the Laughlin states the filling factor, the fractional charge and the fractional statistics are essentially the same number, it is not possible to glean direct evidence for fractional quantum numbers of the excitations directly from the $I - V$ curves at a single point contact beyond the observation that the FQH edges are chiral Luttinger liquids (for a review on FQH tunneling see Ref. [14] and references therein.)

For generic FQH states the situation is more complex and it is not clear if a unique duality exists (Recently[15], a dual picture was proposed for the non-Abelian FQH state at $\nu = 5/2$.) Within the hierarchical description of the FQH edge states (for a review see Ref. [16]), the electron droplet acquires an onion-shell-like structure, with the states lower in the hierarchy laying in the outer regions and the higher states in the inner regions. Given this structure, a point contact can (and does) mix these edges in a complex way. Thus one can envision a series of pinch-off transitions as one layer is peeled-off after another. Clearly, there is no universal electron-quasiparticle duality in this regime.

On the other hand, if the edge of a hierarchical FQH state were to remain sharp down to the scale of the magnetic length, it should not be possible (or meaningful) to physically resolve the many layers assumed by the hierarchies. In this regime the edge should become simple. Such a theory of a simple unreconstructed edge state for the Jain FQH states was presented in Ref. [17]. The main purpose of this paper is to show that for these simple edges there is an uniquely defined electron-quasiparticle duality, representing the pinch-off process. Furthermore, it will turn out that this construction is compatible with a degree of edge reconstruction.

A straightforward extension of duality to the case of the Jain states is hindered by differences in the structure of the Coulomb gas expansion (discussed below) in the form of extra statistical phase factors for the tunneling events around each of the starting fixed points (quasiparticle and electron tunneling). These phase factors spoil the simple weak tunneling *vs.* instanton duality clearly at work in the Laughlin states. Here we show how to use the structure of these expansions to construct a generalized duality for the edge states of the Jain sequence. A key ingredient in the construction of this generalized duality is to account for the correct statistics of the quasiparticles, which is indeed *necessary* in order to reach the

electron tunneling limit (with the correct dual fermionic statistics). To achieve this goal, we combine the edge state formulation for the Jain states in Ref. [17] to recent advances on treating statistical phases in quantum impurity problems in Ref. [18, 19].

The formulation of edge modes of Ref. [17] contains a charged mode, and a neutral topological mode. The real-time Lagrangian is

$$\begin{aligned}\mathcal{S}_{R/L} &= \frac{1}{4\pi}\int dt\,dx\;\partial_x\phi^C_{R,L}\;\left(\mp\partial_t-\partial_x\right)\phi^C_{R,L}\\ &+ \frac{1}{4\pi}\int dt\,dx\;\partial_x\phi^T_{R,L}\;\left(\pm\partial_t\right)\phi^T_{R,L}\,,\end{aligned}\tag{1}$$

where the velocity of the charge mode is set to unit. In real time, the propagators for these charged modes are:

$$\begin{aligned}\langle\phi^C_{R,L}(x,t)\;\phi^C_{R,L}(0,0)\rangle &= -\ln[\delta+i(t\mp x)]\\ &\to -\ln|t\mp x| - i\frac{\pi}{2}\,\mathrm{sgn}(t\mp x)\,,\end{aligned}\tag{2}$$

where we considered $|t|,|x|\gg\delta$, a UV cut-off. The real-time propagators for the topological neutral modes are

$$\langle\phi^T_{R,L}(x,t)\;\phi^T_{R,L}(0,0)\rangle = i\frac{\pi}{2}\,\mathrm{sgn}(t+\varepsilon)\,\mathrm{sgn}(\varepsilon\mp x)\,.\tag{3}$$

where $\varepsilon\to 0_+$ is a regulator such that for $x=0$ or $t=0$ results $\mathrm{sgn}(\epsilon)=1$, ensuring that the propagator is consistent with the equal-time commutation relations of the topological fields, and the statistics of the quasiparticles [20].

Edge quasiparticle and electron operators can be constructed as vertex operators:

$$\Psi_{R/L}(x,t)\sim e^{\pm i\,\gamma_C\;\phi^C_{R,L}(x,t)}\;\;e^{\pm i\,\gamma_T\;\phi^T_{R,L}(x,t)}\,.\tag{4}$$

The charge and statistics of the particles are given by $e^*=e\sqrt{\nu}\,\gamma_C$ and $\theta/\pi=\gamma_C^2-\gamma_T^2$. Hence, for quasiparticles with charge $\nu=p/(2np+1)$, $\gamma_C^{\mathrm{qp}}=\sqrt{\nu}/p$ and $\gamma_T^{\mathrm{qp}}=\sqrt{1+1/p}$, and for electron operators $\gamma_C^{\mathrm{e}}=1/\sqrt{\nu}$ and $\gamma_T^{\mathrm{e}}=\sqrt{p^2+p/\nu^2}$ [17].

Tunneling operators that move one quasiparticle or electron between L and R edges at a tunneling point at $x=0$ can be written as

$$T^{\pm}(t)=\Psi^\dagger_{L,R}\Psi_{R,L}\big|_{x=0}\sim e^{\pm i\,\gamma_C\;\phi^C(t)}\;\;e^{\pm i\,\gamma_T\;\phi^T(t)}\,,\tag{5}$$

with $\phi^{C,T}(t)=\phi^{C,T}_R(0,t)+\phi^{C,T}_L(0,t)$. It is convenient at this point to switch to imaginary-time, in which case the propagators at $x\to 0$ become

$$\langle\phi^C(\tau)\;\phi^C(0)\rangle = -\ln|\tau|^2\tag{6}$$
$$\langle\phi^T(\tau)\;\phi^T(0)\rangle = +i\pi\,\mathrm{sgn}(\tau)\tag{7}$$

Tunneling process enter in the action through a term of the form

$$\mathcal{S}_{\mathrm{tun}}=\int d\tau\;\left[\Gamma\;T^+(\tau)+\Gamma^*\;T^-(\tau)\right]\,.\tag{8}$$

where Γ is the tunneling amplitude and Γ^* its complex conjugate. A perturbative series in the tunneling amplitude for quasiparticles corresponds to a Coulomb gas expansion using insertions of $T^{\pm}(\tau_i)$ at times labeled by τ_i, containing, at each order in the expansion of the partition function, terms such as

$$\begin{aligned}&\langle T^{q_n}(\tau_n)\;\ldots\;T^{q_2}(\tau_2)\;T^{q_1}(\tau_1)\rangle =\\ &\delta\big(\textstyle\sum_j q_j\big)\;\exp\Bigg[\gamma_C^2\sum_{j>k}q_j\,q_k\ln|\tau_j-\tau_k|^2\\ &\qquad -i\pi\;\gamma_T^2\sum_{j>k}q_j\,q_k\;\mathrm{sgn}(\tau_j-\tau_k)\Bigg]\,,\end{aligned}\tag{9}$$

where $q_i=\pm1$ is the charge associated with each vertex operator that is inserted. The $\sum_j q_j=0$ is the neutrality condition of the Coulomb gas expansion (non-neutral terms give vanishing expectation values).

An important feature of the expansion presented in Eq.(9) is that, even in the imaginary-time formulation, there are pure phase factors coming from the contributions due to the topological modes. In the Laughlin sequence, only the real terms (those with the logs) are present, and the weak-strong tunneling duality simply corresponds to an electric/magnetic charge duality [12]. In the Jain sequence, the extra phase factors change order by order the prefactors in the perturbative expansion of the partition function, and consequently change the nature of the strong fixed point reached at large tunneling amplitude Γ. The situation is similar to that found in Ref. [18, 19], where phase factors in the Coulomb gas expansion due to fermionic statistics alter qualitatively the strong coupling limit.

In order to take into account the phase factors due to the topological modes in Eq. (9), we construct a different representation of the problem which gives, order by order, the same factors in the Coulomb gas expansion. In this alternative representation the weak-strong duality becomes apparent. The same phase factors in Eq. (9) can be accounted for in the following quantum impurity problem, which involves a bosonic φ-field and its conjugate momentum, the dual θ-field. Both fields enter when writing the vertex operators that represent the tunneling operators. Consider the tunneling operator

$$\tilde{T}^q(\tau,x)\sim e^{iq\,\alpha\varphi(\tau,x)}\;\;e^{iq\,\beta\theta(\tau,x)},\tag{10}$$

where $q=\pm1$ is the charge associated with this composite vertex operator. The correlation function of n such operators in the Coulomb gas expansion is

$$\langle\tilde{T}^{q_n}(\tau_n,x_n)\;\ldots\;\tilde{T}^{q_2}(\tau_2,x_2)\;\tilde{T}^{q_1}(\tau_1,x_1)\rangle.\tag{11}$$

In evaluating the expression above, we will take the coordinate x to play the role of time (we will use x-ordering, $x_1<x_2<\cdots<x_n$, and all of these $\to 0$), and take τ to

be in the space direction, so that the canonical commutation relation reads

$$[\theta(\tau), \varphi(\tau')] = i\pi \,\mathrm{sgn}(\tau - \tau'). \tag{12}$$

Now, move all the exponentials of θ to the right, using the commutation relations. Each time one moves an exponential of θ past an exponential of φ there is an extra phase factor. The total phase accumulated is

$$e^{-i\pi\,\alpha\beta \sum_{j>k} q_j\, q_k\, \mathrm{sgn}(\tau_j - \tau_k)}. \tag{13}$$

Once moved to the right, the exponentials of θ act like the identity operator, *i.e.* they give a factor of 1, when applied to a state $|D\rangle$ with Dirichlet boundary condition (BC) on θ. This is so since $\theta|D\rangle = \theta_0|D\rangle$, where θ_0 is a constant, and the neutrality of the Coulomb gas ensures that exponentials of all the θ's at different times multiply to 1.

The remaining correlation functions of the exponentials of φ can be calculated with respect to the state $|D\rangle$, and together with the extra phase factors, give:

$$\begin{aligned}&\langle \tilde{T}^{q_1}(\tau_1, x_1)\, \tilde{T}^{q_2}(\tau_2, x_2)\ \ldots\ \tilde{T}^{q_n}(\tau_n, x_n)\rangle_D = \\ &\delta(\textstyle\sum_j q_j)\ \exp\Bigg[\alpha^2 \sum_{j>k} q_j\, q_k \ln|\tau_j - \tau_k|^2 \\ &\qquad -i\pi\,\alpha\beta \sum_{j>k} q_j\, q_k\, \mathrm{sgn}(\tau_j - \tau_k)\Bigg]. \end{aligned} \tag{14}$$

A similar calculation can be carried for the case in which θ obeys a Neumann boundary condition (or Dirichlet on φ, *i.e.*, $\varphi|N\rangle = \varphi_0|N\rangle$). In this case, the exponentials of φ that were moved to the left give a factor of 1 when applied to the $\langle N|$ state. Such calculation gives:

$$\begin{aligned}&\langle \tilde{T}^{q_1}(\tau_1, x_1)\, \tilde{T}^{q_2}(\tau_2, x_2)\ \ldots\ \tilde{T}^{q_n}(\tau_n, x_n)\rangle_N = \\ &\delta(\textstyle\sum_j q_j)\ \exp\Bigg[\beta^2 \sum_{j>k} q_j\, q_k \ln|\tau_j - \tau_k|^2 \\ &\qquad -i\pi\,\alpha\beta \sum_{j>k} q_j\, q_k\, \mathrm{sgn}(\tau_j - \tau_k)\Bigg]. \end{aligned} \tag{15}$$

We can now compare the expansions for the tunneling problem of Jain quasiparticles and electrons in Eq. (9) to those in Eqs. (14,15). We obtain an identity if we take

$$\alpha = \frac{\sqrt{\nu}}{p} \qquad \beta = \frac{1}{\sqrt{\nu}}\,, \tag{16}$$

for which we obtain a quasiparticle-electron duality.

Dirichlet BC: the quasiparticle. With the state $|D\rangle$, the scaling dimension (the prefactor of the log) is $(\gamma_C^{\rm qp})^2 = \alpha^2 = \nu/p^2$. The extra phase factor is $-\pi\alpha\beta = -\pi\frac{1}{p}$. These factors must be compared to the phase factors in Eq. (9), which are $-\pi(\gamma_T^{\rm qp})^2 = -\pi\left(\frac{1}{p}+1\right)$. The extra $-\pi \times 1$ can be simply obtained by using an extra Majorana fermion in front of the $\tilde{T}$ operator. Thus, we match the series expansion in Eq. 14 for the case of D BC on the field φ with the series expansion for the quasiparticle tunneling problem.

Neumann BC: the electron. With the state $|N\rangle$, the scaling dimension (the prefactor of the log) is $(\gamma_C^{\rm e})^2 = \beta^2 = 1/\nu$. The extra phase factor is $-\pi\alpha\beta = -\pi\frac{1}{p}$. One thus matches the result from the expansion in the case of electron tunneling, in which case the phase factors are $-\pi(\gamma_T^{\rm e})^2 = -\pi\frac{p^2+p}{\nu^2} \equiv -\pi\left(\frac{1}{p}+1\right) \pmod{2\pi}$. Again, we are just missing an extra $\pi \times 1$ that can be simply obtained by using an extra Majorana fermion in front of the T operator.

What we learn from the mapping between the tunneling problem for the Jain states and the auxiliary quantum impurity problem is that the extra phase factor from the topological modes introduced in Ref. [17] can be accounted for by keeping both the field φ and its conjugate momentum θ in the vertex operator. Once one chooses the correct vertex operators with both φ and θ, duality amounts to swapping the boundary conditions N and D.

Thus, we have constructed an explicit strong tunneling-weak tunneling duality transformation for generic states in the Jain sequence. The transformation exchanges the quasiparticle and the electron, and Dirichlet with Neumann boundary conditions, and can be summarized by the mapping $\frac{\sqrt{\nu}}{p} \leftrightarrow \frac{1}{\sqrt{\nu}}$. It is consistent with the well known duality for the Laughlin states ($p = 1$).

All of the discussion above concerns a sharp edge, containing simply a charge mode and a topological mode. A weakly reconstructed edge will contain additional neutral modes. (We will not discuss here the generic reconstructed case which is highly non-universal.) These neutral modes, and the local details of the contact, alter the scaling dimentions of the tunneling operators. These changes must be balanced by additional phases in order to provide the correct statistics at a given starting fixed point. Indeed, two interesting question are the following: 1) How to dualize a theory once neutral modes are added, given that the correct statistics are fixed in the starting point? 2) Does the dualized theory contains quasiparticles with the correct statistics at the other endpoint? We answer both questions below.

Generically, for tunneling through a single point (single impurity problem), the neutral modes and the charge mode can be consolidated together into a single field $\phi^P(t)$. This field encapsulates (for the purpose of tunneling) all the *propagating* modes, including the effects of interactions between them. The new tunneling operator can be obtained by replacing $\phi^C \rightarrow \phi^P$ with $\gamma_C^2 \rightarrow \gamma_P^2 = (1+\epsilon)\,\gamma_C^2$ into Eq. (5) for $T^{\pm}(t)$. Notice that the scaling dimension of the tunneling operator is changed to $(1+\epsilon)\,\gamma_C^2$.

The statistical angle of the tunneling quasiparticles is:

$$\theta/\pi = \gamma_C^2\,(1+\epsilon) - \gamma_T^2, \tag{17}$$

which requires picking a coefficient for the topological terms such that $\epsilon\,\gamma_C^2 - \gamma_T^2 = -\bar{\gamma}_T^2$, with $\bar{\gamma}_T$ the original coefficient in the absence of the neutral modes. Such choice will now yield, in imaginary time, terms such as

$$\begin{aligned}
&\langle T^{q_n}(\tau_n)\ \ldots\ T^{q_2}(\tau_2)\, T^{q_1}(\tau_1)\rangle = \\
&\delta(\textstyle\sum_j q_j)\ \exp\Bigg[\gamma_C^2\,(1+\epsilon)\displaystyle\sum_{j>k} q_j\, q_k \ln|\tau_j-\tau_k|^2 \\
&\qquad -i\pi\,\epsilon\,\gamma_C^2 \sum_{j>k} q_j\, q_k\ \mathrm{sgn}(\tau_j-\tau_k) \\
&\qquad -i\pi\,\bar{\gamma}_T^2 \sum_{j>k} q_j\, q_k\ \mathrm{sgn}(\tau_j-\tau_k)\Bigg]
\end{aligned} \tag{18}$$

in the Coulomb gas expansion of tunneling events. The third line in Eq. (18) is the phase that we can represent in terms of Eq. (13), which originated from moving the φ fields leftward and the θ fields rightward, past one another, in the reordering of the vertex operators. It is the first and second lines in Eq. (18) that has to be dealt with now. These two terms together can be accounted for if the equivalent dissipative quantum mechanics action $\mathcal{S}_{\rm diss}$ for the ϕ field is changed as follows:

$$\frac{1}{4\pi}\int\! d\omega|\omega|\ |\varphi(\omega)|^2 \to \frac{1}{4\pi}\int\! d\omega\ (A|\omega| + B\omega)\ |\varphi(\omega)|^2\ . \tag{19}$$

With such dissipative action, one obtains the following correlation functions for the φ and θ fields:

$$\langle\varphi(\tau)\,\varphi(0)\rangle = -\frac{A}{A^2-B^2}\ln|\tau|^2 + i\pi\frac{B}{A^2-B^2}\mathrm{sgn}(\tau) \tag{20}$$

$$\langle\theta(\tau)\,\theta(0)\rangle = -A\ln|\tau|^2 - i\pi B\ \mathrm{sgn}(\tau)\ . \tag{21}$$

Starting from quasiparticle tunneling, one can match the Coulomb gas expansion in Eq. (18) by choosing the same α, β as before [which ensures that the third line in Eq. (18) remains the same], and

$$\frac{A}{A^2-B^2} = 1+\epsilon^{\rm qp}\ ,\quad \frac{B}{A^2-B^2} = \epsilon^{\rm qp}\ . \tag{22}$$

Notice that from these relations it follows that $A+B=1$, *regardless* of the contribution from the neutral modes. This is *very important*, since the statistics of the particle that tunnels in the dual picture (the electron) can be obtained using the θ correlators to be

$$\frac{\theta^e}{\pi} = \beta^2\,(A+B) - \alpha\beta - 1 \tag{23}$$

(the -1 is due to the Majorana modes) which is then *independent* of the neutral modes added to the dual quasiparticle tunneling fixed point. For completeness, let us list the relations between the A, B coefficients and the electron parameters:

$$A = 1+\epsilon^{\rm e}\ ,\quad -B = \epsilon^{\rm e}\ , \tag{24}$$

from which it follows a relation $(\epsilon^{\rm e})^{-1} + (\epsilon^{\rm qp})^{-1} = -2$. In turn, this constraints the relation between the exponents of the $I-V$ tunneling characteristics at the two regimes:

$$I \propto V^{2g-1} \tag{25}$$

where $g = \gamma_C^2(1+\epsilon)$. The exponents associated with quasiparticle and electron tunneling satisfy

$$\frac{1/\nu}{g^{\rm e}} + \frac{\nu/p^2}{g^{\rm qp}} = 2\ . \tag{26}$$

[Notice that in the absence of neutral modes, $g^{\rm e} = \nu^{-1}$ and $g^{\rm qp} = \nu/p^2$, and Eq. (26) is trivially satisfied.]

We presented a construction of a quasiparticle-electron duality valid for generic states of the Jain sequences of FQH states with sharp unreconstructed edges. we showed that in this case too there is a strong tunneling-weak tunneling duality. A key ingredient of this nonperturbative construction is the crucial role played by fractional statistics as a consistency condition. The weak-to-strong tunneling crossover strongly reflects the fractional statistics of the FQH quasiparticles. This mapping also applies for the case of a weakly reconstructed edge.

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[1] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[2] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[3] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[4] C. de C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X.-G. Wen, Phys. Rev. B **55**, 2331 (1997).
[5] I. Safi, P. Devillard, and T. Martin, Phys. Rev. Lett. **86**, 4628 (2001).
[6] C. L. Kane, Phys. Rev. Lett. **90**, 226802 (2003).
[7] S. Vishveshwara, Phys. Rev. Lett. **91**, 196803 (2003).
[8] E.-A. Kim, M. Lawler, S. Vishveshwara, and E. Fradkin, Phys. Rev. Lett. **95**, 176402 (2005).
[9] E. Kim (2006), unpublished; arXiv: cond-mat/0604359.
[10] F. E. Camino, W. Zhou, and V. J. Goldman, Phys. Rev. B **72**, 075342 (2005).
[11] X.-G. Wen, Phys. Rev. B **44**, 5708 (1991).
[12] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **46**, 15233 (1992).
[13] P. Fendley, A. W. W. Ludwig, and H. Saleur, Phys. Rev. Lett. **74**, 3005 (1995).
[14] A. M. Chang, Rev. Mod. Phys. **75**, 1449 (2003).
[15] P. Fendley, M. P. A. Fisher, and C. Nayak, Phys. Rev. Lett. **97**, 036801 (2006).

[16] X. G. Wen, Adv. Phys. **44**, 405 (1995).
[17] A. López and E. Fradkin, Phys. Rev. B **59**, 15323 (1999); *ibid* **63**, 085306 (2001).
[18] C. Chamon, M. Oshikawa, and I. Affleck, Phys. Rev. Lett. **91**, 206403 (2003).
[19] M. Oshikawa, C. Chamon, and I. Affleck, J. Stat. Mech. **0602**, P008 (2005).
[20] The Lagrangian for the topological fields, the second term of Eq. (1), dictates their equal time commutation relations. It has a huge symmetry as it is invariant under the shift of the fields by arbitrary functions of time and space separately. As such, it is not invertible, and the propagator of Eq. (3) is defined by a set of boundary conditions fixed by the regulator ϵ, which ensures the consistency with the commutation relations.